



TITLE:

A Language Equation and Its Applications (Algorithms in Algebraic Systems and Computation Theory)

AUTHOR(S):

Ito, Masami

CITATION:

Ito, Masami. A Language Equation and Its Applications (Algorithms in Algebraic Systems and Computation Theory). 数理解析研究所講究録 2002, 1268: 7-10

ISSUE DATE:

2002-06

URL:

<http://hdl.handle.net/2433/42128>

RIGHT:

A Language Equation and Its Applications

京都産業大学・理学部 伊藤 正美

Masami Ito

Department of Mathematics

Kyoto Sangyo University

Kyoto 603-8555, Japan

Email: ito@ksuvsx0.kyoto-su.ac.jp

Let $u, v \in X^*$ be words over an alphabet X . Then the set $\{u_1v_1u_2v_2 \dots u_nv_n \mid u = u_1u_2 \dots u_n, v = v_1v_2 \dots v_n, u_1, v_1, u_2, v_2, \dots, u_n, v_n \in X^*, n \geq 1\}$ is called the *shuffle product* of u and v , and denoted by $u \diamond v$. For languages $A, B \subseteq X^*$, the set $A \diamond B = \bigcup_{u \in A, v \in B} u \diamond v$ is called the *shuffle product* of A and B . In this paper, we consider the following problem: Let $A, B \subseteq X^*$ be regular languages. Then can we obtain a solution $C \subseteq X^*$ of the language equation $A = B \diamond C$? Obviously, this problem is equivalent to the shuffle decomposition problem for regular languages. Regarding definitions and notations concerning formal languages and automata, not defined in this paper, refer, for instance, to [1].

Now let $\mathcal{A} = (S, X, \delta, s_0, F)$ be a finite automaton with $\mathcal{T}(\mathcal{A}) = A$ and let $\mathcal{B} = (T, X, \gamma, t_0, G)$ be a finite automaton with $\mathcal{T}(\mathcal{B}) = B$. We will look for a regular language C over X such that $A = B \diamond C$. By \overline{X} , we denote the language $\{\bar{a} \mid a \in X\}$ with $X \cap \overline{X} = \emptyset$. Let $\overline{\mathcal{B}} = (T, X \cup \overline{X} \cup \{\#\}, \overline{\gamma}, t_0, G)$ where $\overline{\gamma}$ is defined as follows:

For $t \in T$ and $a \in X$, $\overline{\gamma}(t, a) = t$, $\overline{\gamma}(t, \bar{a}) = \gamma(t, a)$. Moreover, $\overline{\gamma}(t, \#) = t$ if $t \in G$.

Then the following can be easily shown.

Fact 1 Let $a_1a_2 \dots a_n \in X^*$ where $a_i \in X, i = 1, 2, \dots, n$. Then $a_1a_2 \dots a_n \in \mathcal{T}(\mathcal{B})$ if and only if $u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\# \in \mathcal{T}(\overline{\mathcal{B}})$ where $u_1, u_2, \dots, u_n \in X^*$.

Let $\mathcal{A}_1 = (\overline{S}, X \cup \overline{X} \cup \{\#\}, \overline{\delta}, s_0, \{\alpha, \omega\})$ and let $\mathcal{A}_2 = (\overline{S}, X \cup \overline{X} \cup \{\#\}, \overline{\delta}, s_0, \{\alpha\})$ where $\overline{S} = (\bigcup_{a \in X \cup \{\epsilon\}} S^{(a)}) \cup \{\alpha, \omega\}$. Here $S^{(\epsilon)}$ is regarded as S where ϵ is the empty word. For $s \in S, t \in S \setminus F, t' \in F, a \in X \cup \{\epsilon\}, b \in X$ and $\{\#\}$, $\overline{\delta}$ is defined as follows:

$$\bar{\delta}(s^{(a)}, b) = \delta(s, b)^{(a)}, \bar{\delta}(s^{(a)}, \bar{b}) = \delta(s, b)^{(b)}, \bar{\delta}(t^{(a)}, \#) = \{\alpha\} \text{ and } \bar{\delta}(t'^{(a)}, \#) = \{\omega\}.$$

We consider the following two automata:

$$\mathcal{C}_1 = (\bar{S} \times T, X \cup \bar{X} \cup \{\#\}, \bar{\delta} \times \bar{\gamma}, (s_0, t_0), \{\alpha, \omega\} \times G), \mathcal{C}_2 = (\bar{S} \times T, X \cup \bar{X} \cup \{\#\}, \bar{\delta} \times \bar{\gamma}, (s_0, t_0), \{\alpha\} \times G) \text{ where } \bar{\delta} \times \bar{\gamma}((\bar{s}, t), a) = (\bar{\delta}(\bar{s}, a), \bar{\gamma}(t, a)) \text{ for } (\bar{s}, t) \in \bar{S} \times T \text{ and } a \in X.$$

Now consider the following homomorphism ρ of $(X \cup \bar{X} \cup \{\#\})^*$ into X^* :

$$\rho(a) = a \text{ for } a \in X, \rho(\bar{a}) = \epsilon \text{ for } a \in X \text{ and } \rho(\#) = \epsilon.$$

Lemma 1 *Automata accepting the languages $\rho(\mathcal{T}(\mathcal{C}_1))$ and $\rho(\mathcal{T}(\mathcal{C}_2))$ can be effectively constructed.*

Proof Let $i = 1, 2$. From \mathcal{C}_i , we can construct a regular grammar \mathcal{G}_i such that $\mathcal{L}(\mathcal{G}_i) = \mathcal{T}(\mathcal{C}_i)$ with the production rules of the form $A \rightarrow aB$ (A, B are variables and $a \in X \cup \bar{X} \cup \{\#\}$). Replacing every rule of the form $A \rightarrow aB$ in \mathcal{G}_i by $A \rightarrow \rho(a)B$, we can obtain a new grammar \mathcal{G}'_i . Then it is clear that $\rho(\mathcal{L}(\mathcal{C}_i)) = \mathcal{L}(\mathcal{G}'_i)$. Using this grammar \mathcal{G}'_i , we can construct an automaton \mathcal{D}_i such that $\mathcal{T}(\mathcal{D}_i) = \mathcal{T}(\mathcal{G}'_i)$ i.e. $\rho(\mathcal{T}(\mathcal{C}_i)) = \mathcal{T}(\mathcal{D}_i)$. Notice that all the above procedures are effectively done. This completes the proof of the lemma.

Let $B, C \subseteq X^*$. By $B \diamond C$ we denote the shuffle product of B and C , i.e. $\{u_1v_1u_2v_2 \dots u_nv_n \mid u = u_1u_2 \dots u_n \in B, v = v_1v_2 \dots v_n \in C\}$.

Proposition 1 *Let $u \in X^*$. Then $\{u\} \diamond B \subseteq A$ if and only if $u \in \rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$.*

Proof (\Rightarrow) Let $u = u_1u_2 \dots u_nu_{n+1} \in X^*$ and let $a_1a_2 \dots a_n \in B$ where $u_1, u_2, \dots, u_n, u_{n+1} \in X^*$ and $a_1, a_2, \dots, a_n \in X$. Then $\bar{\delta} \times \bar{\gamma}((s_0, t_0), u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\#) = (\bar{\delta}(s_0, u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\#), \bar{\gamma}(t_0, u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\#)) = (\bar{\delta}(\delta(s_0, u_1a_1u_2a_2 \dots u_na_nu_{n+1})^{(a_n)}, \#), \bar{\gamma}(\gamma(s_0, a_1a_2 \dots a_n)^{(a_n)}, \#)) = (\omega, \gamma(t_0, a_1a_2 \dots a_n)) \in \{\omega\} \times G$. Therefore, $u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\# \in \mathcal{T}(\mathcal{C}_1) \setminus \mathcal{T}(\mathcal{C}_2)$. Hence $u = u_1u_2 \dots u_nu_{n+1} = \rho(u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\#) \in \rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$.

(\Leftarrow) Suppose that $u \diamond B \subseteq A$ does not hold though $u \in \rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$. Then there exist $u = u_1u_2 \dots u_nu_{n+1} \in X^*$ and $a_1a_2 \dots a_n \in B$ such that $u_1a_1u_2a_2 \dots u_na_nu_{n+1} \notin A$. Hence $\bar{\gamma}(t_0, u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\#) = \bar{\gamma}(\gamma(t_0, a_1a_2 \dots a_n), \#) = \gamma(t_0, a_1a_2 \dots a_n) \in G$. On the other hand, since $u_1a_1u_2a_2 \dots u_na_nu_{n+1} \notin A$, we have $\bar{\delta}(s_0, u_1\bar{a}_1u_2\bar{a}_2 \dots u_n\bar{a}_nu_{n+1}\#) = \bar{\delta}(\delta(s_0, u_1a_1u_2a_2 \dots u_na_nu_{n+1})^{(a_n)}, \#) = \{\alpha\}$.

$u_{n+1})^{(a_n)}, \#) = \{\alpha\}$. Hence $\bar{\delta} \times \bar{\gamma}((s_0, t_0), u_1 \bar{a}_1 u_2 \bar{a}_2 \dots u_n \bar{a}_n u_{n+1} \#) \in \{\alpha\} \times G$, i.e. $u_1 \bar{a}_1 u_2 \bar{a}_2 \dots u_n \bar{a}_n u_{n+1} \# \in \mathcal{T}(\mathcal{C}_2)$. Therefore, $u = \rho(u_1 \bar{a}_1 u_2 \bar{a}_2 \dots u_n \bar{a}_n u_{n+1} \#) \in \rho(\mathcal{T}(\mathcal{C}_2))$. On the other hand, it is obvious that $u_1 \bar{a}_1 u_2 \bar{a}_2 \dots u_n \bar{a}_n u_{n+1} \# \in \mathcal{T}(\mathcal{C}_1)$. Thus $u \notin \rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$, a contradiction. Consequently, the proposition must hold true.

Corollary *In the above, $B \diamond (\rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))) \subseteq A$.*

Let $L \subseteq X^*$ be a regular language over X . By $\#L$, we denote the number $\min\{|S| \mid \exists \mathcal{A} = (S, X, \delta, s_0, F), L = \mathcal{T}(\mathcal{A})\}$ where $|S|$ denotes the cardinality of S . Moreover, $\mathcal{I}(n, X)$ denotes the class of languages $\{L \subseteq X^* \mid \#L \leq n\}$.

Theorem 1 *Let $A \subseteq X^*$ and let n be a positive integer. Then it is decidable whether there exist nontrivial regular languages $B \in \mathcal{I}(n, X)$ and $C \subseteq X^*$ such that $A = B \diamond C$. Here a language $D \subseteq X^*$ is said to be nontrivial if $D \neq \{\epsilon\}$.*

Proof Let $A \subseteq X^*$ be a regular language. Assume that there exist nontrivial regular languages $B \in \mathcal{I}(n, X)$ and $C \subseteq X^*$ such that $A = B \diamond C$. Then, by Proposition 1 and its corollary, $C \subseteq \rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$ and $B \diamond (\rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))) \subseteq A$. Hence $A = B \diamond (\rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2)))$. Thus we have the following algorithm: (1) Choose a nontrivial regular language $B \subseteq X^*$ from $\mathcal{I}(n, X)$ and construct the language $\rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$ (see Lemma 1). (2) Let $C = \rho(\mathcal{T}(\mathcal{C}_1)) \setminus \rho(\mathcal{T}(\mathcal{C}_2))$. (3) Compute $B \diamond C$. (4) If $A = B \diamond C$, then the output is "YES" and "NO", otherwise. (4) If the output is "NO", then choose another element in $\mathcal{I}(n, X)$ as B and continue the procedures (1) - (3). (5) Since $\mathcal{I}(n, X)$ is a finite set, the above process terminates after a finite-step trial. Once one gets the output "YES", then there exist nontrivial regular languages $B \in \mathcal{I}(n, X)$ and $C \subseteq X^*$ such that $A = B \diamond C$. Otherwise, there are no such languages.

Let n be a positive integer. By $\mathcal{F}(n, X)$, we denote the class of finite languages $\{L \subseteq X^* \mid \max\{|u| \mid u \in L\} \leq n\}$ where $|u|$ is the length of u . Then the following result by C. Cămpăanu et al. ([2]) can be obtained as a corollary of the above theorem.

Corollary *For a given positive integer n and a regular language $A \subseteq X^*$, the problem whether $A = B \diamond C$ for a nontrivial language $B \in \mathcal{F}(n, X)$ and a nontrivial regular language $C \subseteq X^*$ is decidable.*

Proof Obvious from the fact that $\mathcal{F}(n, X) \subseteq \mathcal{I}(|X|^{n+1}, X)$.

References

- [1] J.E. Hopcroft and J.D. Ullman, *Introduction to Automata Theory, Languages and Computation*, Addison-Wesley, Reading MA, 1979.
- [2] C. Câmpeanu, K. Salomaa, S. Vágvolgyi, Shuffle quotient and decompositions, *Lecture Notes in Computer Science* (Springer), to appear.